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Mermin–Ho vortices and monopoles in three-component spinor BEC

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Abstract

In the light of the ϕ-mapping topological current theory, the Mermin–Ho vortices and the monopoles in three-component spinor BEC are studied. It is pointed out that these two topological structures respectively inhere in two-dimensional and three-dimensional topological currents which can be derived from the same field tensor $k_{\mu\nu}$, and both these topological structures are characterized by the ϕ-mapping topological numbers—Hopf indices and Brouwer degrees. Furthermore, the spatial bifurcation of Mermin–Ho vortices and the generation and annihilation of monopoles are also discussed.

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1. Introduction

The ϕ-mapping topological current theory plays an important role in the study of condensed matter physics [1]. In this paper, using this theory two important topological structures—the Mermin–Ho vortices and the monopoles in three-component spinor BEC—are discussed [2–7]. The physical system we study is the Bose condensates with three internal hyperfine spin states: $\psi_i = |F = 1, m_F = +1, 0, -1\rangle$ $(i = 1, 2, 3)$. This is equivalent to a spin-1 quantum fluid, which is governed by [2, 3]

$$H = \psi_i^* \left(-\frac{\hbar \vec{\nabla}^2}{2m} + V - \mu_i \right) \psi_i + \frac{1}{2} g_\rho \hat{\rho}^2 + \frac{1}{2} g_s \hat{\mathbf{S}} \cdot \hat{\mathbf{S}} \qquad (i = 1, 2, 3). \qquad (1)$$

Here V is the external confinement potential such as an optical potential, and μ_i is the chemical potential; $\hat{\rho} = \psi_i^* \psi_i$ and $\hat{\mathbf{S}}_a = \psi_i^* (S_a)_{ij} \psi_j$ respectively denote the particle number and spin densities, with $g_\rho = \frac{4\pi\hbar^2}{m} \frac{a_0 + 2a_2}{3}$ and $g_s = \frac{4\pi\hbar^2}{m} \frac{a_2 - a_0}{3}$ (a_0 and a_2 are the scattering lengths).

Introducing the vector wavefunction Φ as $\Phi = (\psi_1\psi_2\psi_3)^T$, the unit vector of hyperfine spin is defined as

$$n^a = \frac{\Phi^\dagger S_a \Phi}{\Phi^\dagger \Phi} \qquad (a = 1, 2, 3, n^a n^a = 1) \tag{2}$$

where S_a are the spin matrices:

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

satisfying $[S_a, S_b] = \mathrm{i}\epsilon_{abc} S_c$.

To discuss the Mermin–Ho vortices and the monopoles in three-component spinor BEC, it is necessary to study the field tensor [8]

$$k_{\mu\nu} = \epsilon_{abc} n^a \partial_\mu n^b \partial_\nu n^c \qquad (\mu, \nu = 0, 1, 2, 3) \tag{3}$$

which is a topological term describing the non-uniform distribution of the hyperfine spin unit vector $\vec{n}$ at large distances in space. In this paper, by making use of our ϕ-mapping topological current theory [1, 9–12], the Mermin–Ho vortices [6] and the monopoles [7, 8] in three-component spinor BEC are discussed by studying this $k_{\mu\nu}$ tensor. It is pointed out that there are different kinds of $\vec{n}$ field distributions existing in space which lead to different topological structures. To express these topological structures clearly, some other vector order parameters are defined from $\vec{n}$. Using these new order parameters, it is revealed that there are respectively two-dimensional and three-dimensional topological currents which can be derived from $k_{\mu\nu}$ through the expression (3), and the above two kinds of topological structures are respectively inhering in these two topological currents. It is pointed out that these two structures are characterized by the ϕ-mapping topological numbers, Hopf indices and Brouwer degrees, and their locations and motions can be rigorously determined. Moreover, in this paper the spatial bifurcation of Mermin–Ho vortices and the generation and annihilation of monopoles are also discussed.

2. The Mermin–Ho vortices in three-component spinor BEC

According to [13], in a field tensor $k_{\mu\nu}$ with inner structure (3), there exist Mermin–Ho vortices. In the following, using the ϕ-mapping topological current theory, it is shown that there is a two-dimensional topological current which can be derived from $k_{\mu\nu}$, and the Mermin–Ho vortex structure is just inhering in this topological current.

The $k_{\mu\nu}$ tensor can be re-expressed in an Abelian field tensor form [9, 14]

$$k_{\mu\nu} = \epsilon_{abc} n^a \partial_\mu n^b \partial_\nu n^c = \partial_\mu W_\nu - \partial_\nu W_\mu \tag{4}$$

where W_μ is the Wu–Yang potential

$$W_\mu = \vec{e}_1 \cdot \partial_\mu \vec{e}_2. \tag{5}$$

Here $\vec{e}_1$ and $\vec{e}_2$ are two unit vectors normal to $\vec{n}$; $(\vec{e}_1, \vec{e}_2, \vec{n})$ forms an orthogonal frame:

$$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{n} = \vec{e}_2 \cdot \vec{n} = 0 \qquad \vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{n} \cdot \vec{n} = 1. \tag{6}$$

Consider another two-component vector order parameter in space: $\vec{\phi} = (\phi^1, \phi^2)$, which resides in the plane formed by the unit vectors $\vec{e}_1$ and $\vec{e}_2$ and satisfies

$$e_1^a = \frac{\phi^a}{\|\phi\|} \qquad e_2^a = \epsilon^{ab}\frac{\phi^b}{\|\phi\|} \qquad (\|\phi\|^2 = \phi^a\phi^a; a, b = 1, 2). \tag{7}$$

It can be proved that the expression for $\vec{e}_1$ and $\vec{e}_2$ (7) satisfies the restriction (6). Obviously the zero points of $\vec{\phi}$ are just the two-dimensional singular points of $\vec{e}_1$ and $\vec{e}_2$. Using the $\vec{\phi}$ field, the Wu–Yang potential can be expressed as

$$W_\mu = \epsilon^{ab}\frac{\phi^a}{\|\phi\|}\partial_\mu\frac{\phi^b}{\|\phi\|} \tag{8}$$

and the field tensor $k_{\mu\nu}$ is

$$k_{\mu\nu} = 2\epsilon^{ab}\partial_\mu\frac{\phi^a}{\|\phi\|}\partial_\nu\frac{\phi^b}{\|\phi\|}. \tag{9}$$

According to [1, 9, 10], using $\partial_\mu\frac{\phi^a}{\|\phi\|} = \frac{\partial_\mu\phi^a}{\|\phi\|} + \phi^a\partial_\mu\frac{1}{\|\phi\|}$ and the Green function relation in ϕ-space: $\partial_a\partial_a \ln\|\phi\| = 2\pi\delta^2(\vec{\phi})$ ($\partial_a = \partial/\partial\phi^a$), it can be proved that

$$\epsilon^{ab}\partial_\mu\frac{\phi^a}{\|\phi\|}\partial_\nu\frac{\phi^b}{\|\phi\|} = \epsilon_{\mu\nu\lambda\rho}2\pi\delta^2(\vec{\phi})D^{\lambda\rho}(\phi/x) \qquad (a, b = 1, 2) \tag{10}$$

where $D^{\lambda\rho}(\phi/x) = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$. Then $k_{\mu\nu}$ can be expressed in a δ-function form

$$k_{\mu\nu} = 4\pi\epsilon_{\mu\nu\lambda\rho}\delta^2(\vec{\phi})D^{\lambda\rho}(\phi/x). \tag{11}$$

According to the ϕ-mapping topological current theory [12], the two-dimensional topological current is defined as

$$\tilde{K}^{\mu\nu} = \frac{1}{2\pi}\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ab}\partial_\lambda n^a\partial_\rho n^b = \delta^2(\vec{\phi})D^{\mu\nu}(\phi/x) \tag{12}$$

so it is revealed that there is a two-dimensional topological current existing in $k_{\mu\nu}$:

$$k_{\mu\nu} = 4\pi\epsilon_{\mu\nu\lambda\rho}\tilde{K}^{\lambda\rho}. \tag{13}$$

Defining the spatial components of $\tilde{K}^{\mu\nu}$ as

$$J^i = \tilde{K}^{0i} \qquad (i = 1, 2, 3) \tag{14}$$

we have

$$J^i = \delta^2(\vec{\phi})D^i(\phi/x) = \frac{1}{8\pi}\epsilon^{ijk}k_{jk} \tag{15}$$

where $D^i(\phi/x) = \frac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j\phi^a\partial_k\phi^b$ is the Jacobian vector.

An important conclusion from (15) is

$$J^i \begin{cases} = 0 & (\text{iff } \delta^2(\vec{\phi}) = 0, \vec{\phi} \neq 0) \\ \neq 0 & (\text{iff } \delta^2(\vec{\phi}) \neq 0, \vec{\phi} = 0) \end{cases} \tag{16}$$

so it is necessary to study the zero points of $\vec{\phi}$ to determine the non-zero solutions of J^i. The implicit function theory shows that [15], under the regular condition $D^{\mu\nu}(\phi/x) \neq 0$, the general solutions of

$$\phi^a(x^0 = t, x^1, x^2, x^3) = 0 \qquad (a = 1, 2) \tag{17}$$

can be expressed as

$$x^1 = x^1_j(s, t) \qquad x^2 = x^2_j(s, t) \qquad x^3 = x^3_j(s, t) \qquad (j = 1, 2, \ldots, N) \tag{18}$$

which represent the evolution surfaces of N isolated strings L_j in (3 + 1)-dimensional spacetime with s being the string parameter. These topological string structures are just the Mermin–Ho vortices.

The spatial structure and the evolution of Mermin–Ho vortices can be discussed by making use of the ϕ-mapping theory. Firstly, we study the spatial structure of these vortex lines by fixing the time coordinate t. In δ-function theory [16], one can prove

$$\delta^2(\vec{\phi}) = \sum_{j=1}^{N} \beta_j \int_{L_j} \frac{\delta^3(\vec{x} - \vec{x}_j(s))}{|D(\phi/u)|_{\Sigma_j}} \, \mathrm{d}s \tag{19}$$

where $D\big(\frac{\phi}{u}\big)_{\Sigma j} = \big(\frac{1}{2}\epsilon^{jk}\epsilon_{mn}\frac{\partial\phi^m}{\partial u^j}\frac{\partial\phi^n}{\partial u^k}\big)$, and Σ_j is the jth planar element transversal to L_j with local coordinates (u^1, u^2). The positive integer β_j is the Hopf index of ϕ-mapping, which means that when the point $\vec{x}$ covers the neighbourhood of the zero point $\vec{x}_j$ once, the vector field $\vec{\phi}$ covers the corresponding region in ϕ-space β_j times. Meanwhile, the direction vector of L_j is [1, 9, 10]

$$\left.\frac{\mathrm{d}x^i}{\mathrm{d}s}\right|_{\vec{x}_j} = \left.\frac{D^i(\phi/x)}{D(\phi/u)_{\Sigma_j}}\right|_{\vec{x}_j} \tag{20}$$

which leads to

$$\left.\frac{\mathrm{d}x^1}{\mathrm{d}x^3}\right|_{\vec{x}_j} = \left.\frac{D^1(\phi/x)}{D^3(\phi/x)}\right|_{\vec{x}_j} \qquad \left.\frac{\mathrm{d}x^2}{\mathrm{d}x^3}\right|_{\vec{x}_j} = \left.\frac{D^2(\phi/x)}{D^3(\phi/x)}\right|_{\vec{x}_j}. \tag{21}$$

Therefore from (19) and (20) we find the inner topological structure of J^i:

$$J^i = \sum_{j=1}^{N} \beta_j \eta_j \int_{L_j} \frac{\mathrm{d}x^i}{\mathrm{d}s} \delta^3(\vec{x} - \vec{x}_j(s)) \, \mathrm{d}s \tag{22}$$

where η_j is the Brouwer degree of ϕ-mapping: $\eta_j = \operatorname{sgn} D(\phi/u)_{\vec{x}_j} = \pm 1$. From (22) one can obtain the topological number of vortex line L_j

$$Q_j = \frac{1}{2\pi} \int_{\Sigma_j} \frac{1}{2} k_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j = \int_{\Sigma_j} J^i \, \mathrm{d}\sigma_i = W_j \tag{23}$$

where $W_j = \beta_j \eta_j$ is the winding number of $\vec{\phi}$ around L_j. And the total topological number on surface Σ is

$$Q = \int_{\Sigma} J^i \, \mathrm{d}\sigma_i = \sum_{j=1}^{N} W_j. \tag{24}$$

Secondly, we discuss the evolution of the Mermin–Ho vortex lines. For simplicity we fix the $x^3 = z$ coordinate and take the XOY plane as the cross section, and the intersection lines between the evolution surfaces and the cross section are just the motion curves of vortices. The velocity of the intersection point between L_j and the cross section is given by

$$v_j^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} = \left.\frac{D^i(\phi/x)}{D(\phi/x)}\right|_{\vec{x}_j} \qquad (i = 1, 2) \tag{25}$$

where $D^0(\phi/x) = \epsilon_{ab}\partial_1 n^a \partial_2 n^b$, $D^1(\phi/x) = \epsilon_{ab}\partial_2 n^a \partial_0 n^b$ and $D^2(\phi/x) = \epsilon_{ab}\partial_0 n^a \partial_1 n^b$. Expression (25) just determines the motion of the jth Mermin–Ho vortex.

In the following, we will simply discuss the spatial bifurcation of these Mermin–Ho vortices. From (21) it can be seen that, when the regular condition $D^{\mu\nu}(\phi/x) \neq 0$ $(i = 1, 2, 3)$ is satisfied, the direction of vortex line L_j at $\vec{x}_j$ is definite. When this condition fails, i.e., when

$$D^i\left(\frac{\phi}{x}\right) = 0 \qquad (i = 1, 2, 3) \tag{26}$$

at some points (marked as z_j^{*i}) along L_j, the functional relationship between the coordinates x^1 and x^3, or x^2 and x^3, is not unique in the neighbourhood of z_j^{*i}, and the direction of L_j expressed by $\mathrm{d}x^1/\mathrm{d}x^3$ and $\mathrm{d}x^2/\mathrm{d}x^3$ in (21) is indefinite. Hence these very points z_j^{*i} are called the bifurcation points of the Mermin–Ho vortex lines in three-dimensional space.

According to the ϕ-mapping theory, the Taylor expansion of the solution of (17) in the neighborhood of z_j^{*i} can be generally expressed as $A\big(x^1 - z_j^{*1}\big)^2 + 2B\big(x^1 - z_j^{*1}\big)\big(x^3 - z_j^{*3}\big) + C\big(x^3 - z_j^{*3}\big)^2 + \cdots = 0$, where A, B and C are constants [1, 10]. This leads to

$$A\left(\frac{\mathrm{d}x^1}{\mathrm{d}x^3}\right)^2 + 2B\frac{\mathrm{d}x^1}{\mathrm{d}x^3} + C = 0 \quad \text{or} \quad C\left(\frac{\mathrm{d}x^3}{\mathrm{d}x^1}\right)^2 + 2B\frac{\mathrm{d}x^3}{\mathrm{d}x^1} + A = 0. \tag{27}$$

The solutions of (27) give different branches of the vortex lines at the bifurcation points. In the following four main cases are simply discussed [1]:

Case 1 ($A \neq 0$). For $\Delta = 4(B^2 - AC) > 0$, from (27) we get two different spatial directions at the bifurcation point: $\frac{\mathrm{d}x^1}{\mathrm{d}x^3}\big|_{1,2} = \frac{-B\pm\sqrt{B^2-AC}}{A}$. This is the intersection of two vortex lines of different directions.

Case 2 ($A \neq 0$). For $\Delta = 4(B^2 - AC) = 0$, we get only one direction at the bifurcation point: $\frac{\mathrm{d}x^1}{\mathrm{d}x^3}\big|_{1,2} = -\frac{B}{A}$. This includes three sub-cases: (a) two vortex lines tangentially contact, i.e. tangentially intersect; (b) two vortex lines merge into one line; (c) one vortex line splits into two lines.

Case 3 ($A = 0, C \neq 0$). For $\Delta = 4(B^2 - AC) > 0$, from (27) we have $\frac{\mathrm{d}x^3}{\mathrm{d}x^1}\big|_{1,2} = \frac{-B\pm\sqrt{B^2-AC}}{C} = 0, -\frac{2B}{C}$. This includes two sub-cases: (a) three vortex lines merge into one line; (b) one vortex line splits into three lines.

Case 4 ($A = C = 0$). (27) gives, respectively, $\frac{\mathrm{d}x^1}{\mathrm{d}x^3} = 0$, $\frac{\mathrm{d}x^3}{\mathrm{d}x^1} = 0$. This case shows that two curves normally intersect at the bifurcation point, which is similar to case 3.

It should be noted that, since the topological current J^i defined in (14) satisfies the continuity equation $\partial_i J^i = 0$, the sum of the topological charge of final vortex line(s) should be equal to that of the initial line(s) at the bifurcation point for a fixed index j: $\sum_f \beta_{j_f}\eta_{j_f} = \sum_i \beta_{j_i}\eta_{j_i}$, where '$i$' stands for 'initial' and 'f' stands for 'final'.

3. The monopoles in three-component spinor BEC

In this section, it is shown that there is also a three-dimensional topological current which can be derived from $k_{\mu\nu}$ when $k_{\mu\nu}$ is expressed within another different order parameter configuration, and the monopoles are just inhering in this three-dimensional topological current.

The generalized winding number W can be obtained by integrating $k_{\mu\nu}$ on a closed surface $\partial\Omega$, where Ω is a spatial volume and $\partial\Omega$ is its boundary. W is defined by the Gauss map $n : \partial\Omega \to S^2$ [9]

$$W = \frac{1}{8\pi}\int_{\partial\Omega} n^*(\epsilon_{abc}n^a\,\mathrm{d}n^b \wedge \mathrm{d}n^c) \tag{28}$$

i.e.

$$W = \frac{1}{8\pi}\int_{\partial\Omega} \epsilon_{abc}n^a\partial_i n^b\partial_j n^c\,\mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{1}{8\pi}\int_{\partial\Omega} k_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \qquad (i, j = 1, 2, 3). \tag{29}$$

In topology this means that when $\vec{x}$ covers $\partial\Omega$ in the real space once, the unit vector $\vec{n}$ will cover S^2 W times. This is a topological invariant and is called the degree of Gauss map.

On the other hand, W is also the total topological charge of the point defects (i.e. the singular points of $\vec{n}$) located in the volume Ω. Using Stokes theorem, we have

$$W = \frac{1}{8\pi}\int_{\Omega} \epsilon^{ijk}\epsilon_{abc}\partial_i n^a \partial_j n^b \partial_k n^c \, \mathrm{d}^3x = \int_{\Omega} \rho \, \mathrm{d}^3x \tag{30}$$

where the density of point defects is derived from k_{ij},

$$\rho = \frac{1}{8\pi}\epsilon^{ijk}\epsilon_{abc}\partial_i n^a \partial_j n^b \partial_k n^c = \frac{1}{8\pi}\epsilon^{ijk}\partial_i k_{jk}. \tag{31}$$

According to the ϕ-mapping topological current theory, the three-dimensional topological current is defined as [12]

$$J^{\mu} = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda\rho}\epsilon_{abc}\partial_\nu n^a \partial_\lambda n^b \partial_\rho n^c \qquad (\mu = 0, 1, 2, 3) \tag{32}$$

so the density ρ is just the temporal component of J^{μ}: $J^0 = \rho$. Therefore it is revealed that there is a three-dimensional topological current which can be derived from $k_{\mu\nu}$. It can be easily seen that the topological current J^{μ} is identically conserved: $\partial_\mu J^\mu = 0$, i.e.,

$$\partial_t \rho + \partial_i J^i = 0. \tag{33}$$

According to [1, 9, 10], it can be proved that

$$J^{\mu} = \delta^3(\vec{\varphi}) D^{\mu}(\varphi/x) \tag{34}$$

i.e., the temporal and spatial components of J^{μ} are respectively

$$\rho = \delta^3(\vec{\varphi}) D(\varphi/x) \qquad J^i = \delta^3(\vec{\varphi}) D^i(\varphi/x) \tag{35}$$

where φ^a is a three-component vector order parameter defined as

$$n^a = \frac{\varphi^a}{\|\varphi\|} \qquad (\|\varphi\|^2 = \varphi^a \varphi^a, a = 1, 2, 3). \tag{36}$$

Obviously the zero points of the $\vec{\varphi}$ field are just the three-dimensional singular points of the $\vec{n}$ field. $D^{\mu}(\varphi/x)$ is the vector Jacobian: $\epsilon^{abc}D^{\mu}(\varphi/x) = \epsilon^{\mu\nu\lambda\rho}\partial_\nu\varphi^a\partial_\lambda\varphi^b\partial_\rho\varphi^c$, and $D(\varphi/x) = D^0(\varphi/x)$.

An important conclusion from (34) is

$$J^{\mu} \begin{cases} = 0 & (\text{iff } \delta^2(\vec{\varphi}) = 0, \vec{\varphi} \neq 0) \\ \neq 0 & (\text{iff } \delta^2(\vec{\varphi}) \neq 0, \vec{\varphi} = 0) \end{cases} \tag{37}$$

so it is necessary to study the zero points of $\vec{\varphi}$ to determine the non-zero solutions of J^{μ}. The implicit function theory shows that [15], under the regular condition

$$D^0(\varphi/x) \neq 0 \tag{38}$$

the general solutions of

$$\varphi^a(x^0 = t, x, y, z) = 0 \qquad (a = 1, 2, 3) \tag{39}$$

can be expressed as

$$x^1 = x_l^1(t) \qquad x^2 = x_l^2(t) \qquad x^3 = x_l^3(t) \qquad (l = 1, 2, \ldots, M) \tag{40}$$

which represent the worldlines of M moving isolated singular points P_l. These singular points are just the monopoles [8, 9].

In δ-function theory [16], one can prove that

$$\delta^3(\vec{\varphi}) = \sum_{l=1}^{M} \frac{\beta_l \eta_l}{D(\varphi/x)_{\vec{x}_l}} \delta^3(\vec{x} - \vec{x}_l) \tag{41}$$

where the positive integer β_l is the Hopf index, and η_l is the Brouwer degree: $\eta_l = \operatorname{sgn} D(\varphi/x)_{\vec{x}_l} = \pm 1$. From (35) and (41), we respectively obtain

$$\rho = \sum_{l=1}^{M} \beta_l \eta_l \delta^3(\vec{x} - \vec{x}_l) \tag{42}$$

$$J^i = \sum_{l=1}^{M} \beta_l \eta_l \delta^3(\vec{x} - \vec{x}_l) \left. \frac{D^i(\varphi/x)}{D(\varphi/x)} \right|_{\vec{x}_l} \qquad (i = 1, 2, 3). \tag{43}$$

The expression for ρ (42) just leads to that for the topological charges of monopoles. Using (30) and (42) we get the total winding number on $\partial\Omega$ around the M monopoles:

$$W = \sum_{l=1}^{M} W_l = \sum_{l=1}^{M} \beta_l \eta_l \tag{44}$$

where W_l is the winding number of $\vec{\varphi}$ around the lth monopole P_l, which means that when $\vec{x}$ covers the boundary of the neighbourhood of P_l in three-dimensional space once, the unit vector $\vec{n}$ will cover S^2 W_l times.

The expression for J^i (43) leads to the description of monopole motion. The velocity of the lth monopole P_l is defined as

$$v_l^i = \frac{\mathrm{d}x_l^i}{\mathrm{d}t} = \left. \frac{D^i(\varphi/x)}{D(\varphi/x)} \right|_{\vec{x}_l} \qquad (i = 1, 2, 3) \tag{45}$$

so the topological current can be written in the same form as the current density in hydrodynamics:

$$J^i = \sum_{l=1}^{M} \beta_l \eta_l \delta^3(\vec{x} - \vec{x}_l) \frac{\mathrm{d}x_l^i}{\mathrm{d}t}. \tag{46}$$

In the following, we will simply discuss the generation and the annihilation of these monopoles at the limit points in spacetime [1]. The limit point $\left(z_l^{*i}, t_l^*\right)$ is defined as

$$D\left(\frac{\varphi}{x}\right)\Big|_{(z_l^{*i}, t_l^*)} = 0 \qquad D^1\left(\frac{\varphi}{x}\right)\Big|_{(z_l^{*i}, t_l^*)} \neq 0 \tag{47}$$

or

$$D\left(\frac{\varphi}{x}\right)\Big|_{(z_l^{*i}, t_l^*)} = 0 \qquad D^2\left(\frac{\varphi}{x}\right)\Big|_{(z_l^{*i}, t_l^*)} \neq 0. \tag{48}$$

Without loss of generality, we only consider the case (47). From expression (45) it can be seen that, at the limit point $\left(z_l^{*i}, t_l^*\right)$, the regular condition (38) fails and the velocity v_l^i is indefinite: $\mathrm{d}x^i/\mathrm{d}t = \infty$ $(i = 1, 2, 3)$. In order to determine the velocity of P_l at $\left(z_l^{*i}, t_l^*\right)$, we can use $D^1(\varphi/x)$ instead of $D(\varphi/x)$ to apply the implicit function theorem. In this case, we have

$$\left. \frac{\mathrm{d}t}{\mathrm{d}x^1} \right|_{(z_l^{*i}, t_l^*)} = \left. \frac{D(\varphi/x)}{D^1(\varphi/x)} \right|_{(z_l^{*i}, t_l^*)} = 0. \tag{49}$$

Taking x^1 as the parameter, we have a unique solution of (39) in the neighbourhood of $\left(z_l^{*i}, t_l^*\right)$,

$$t = t(x^1) \qquad x^2 = x^2(x^1) \qquad x^3 = x^3(x^1) \tag{50}$$

and the Taylor expansion of (50) at $\left(z_l^{*i}, t_l^*\right)$ is

$$t - t_l^* = \frac{1}{2} \left. \frac{\mathrm{d}^2 t}{(\mathrm{d}x^1)^2} \right|_{(z_l^{*i}, t_l^*)} \left(x^1 - z_l^{*1}\right)^2 \tag{51}$$

which is a parabola in the x^1-t plane. From (51), we can obtain the two solution branches $x^1_{\mathrm{I}}(t)$ and $x^1_{\mathrm{II}}(t)$ of (39), which give the evolution lines of two monopoles (respectively marked with I and II): at the limit point $\left(z_l^{*i}, t_l^*\right)$, if $\mathrm{d}^2t/(\mathrm{d}x^1)^2 > 0$, we have the branch process of $t > t^*$, which just describes the generation of two monopoles I and II; otherwise, we have the branch process of $t < t^*$, which just describes the annihilation of two monopoles I and II [1]. It should be pointed out that, since the conservation law (33) is satisfied, the sum of the topological charges of these two monopoles is identically conserved at $\left(z_l^{*i}, t_l^*\right)$: $\beta_{l\mathrm{I}}\eta_{l\mathrm{I}} + \beta_{l\mathrm{II}}\eta_{l\mathrm{II}} = 0$.

4. Conclusion

In this paper, using the ϕ-mapping topological current theory, the Mermin–Ho vortices and the monopoles in three-component spinor BEC are discussed by studying the field tensor $k_{\mu\nu}$. Noting that there are different kinds of $\vec{n}$ field configurations existing in space, we respectively define the vector order parameters $\vec{\phi}$ in section 2 and $\vec{\varphi}$ in section 3; it should be stressed that the difference between $\vec{\phi}$ and $\vec{\varphi}$ just originates from the different configurations of vector $\vec{n}$. Using these two new order parameters, it is revealed that there are two-dimensional and three-dimensional topological currents which can be derived from $k_{\mu\nu}$ through expression (3), and the above two kinds of topological structures are respectively inhering in these two topological currents. It is shown that these two structures are characterized by the ϕ-mapping topological numbers: Hopf indices and Brouwer degrees, and their locations and motions can be rigorously determined. Moreover, in this paper the spatial bifurcation of Mermin–Ho vortices and the generation and annihilation of monopoles are also discussed.

Finally, we point out that in this paper the vortex lines are treated as geometric lines, i.e., the width of a vortex line is zero; but in experiments, this width does not vanish. Then, in experiments, since the Mermin–Ho vortices and the monopoles originate from the non-trivial $\vec{n}$ field distributions at large distances, the width of the 'core' of a Mermin–Ho vortex should be larger than that of a velocity field vortex of a single condensate [1]. For this point and other properties of these topological excitations, especially their energies, see [17] and references therein.

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